

Generalized Chatterjea Type Contractions on Integrated Matrix Graph Metric Spaces

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ABSTRACT

This paper proposes a computationally verifiable hybrid fixed-point framework on the integrated metric space (Z, d_α) , where $Z = Y \times V$ combines a continuous component endowed with the matrix-induced metric $d_M(x, y) = \|M(x - y)\|$ with invertible M and a discrete component (V, d_G) defined by the shortest-path metric of a finite weighted graph. The objective is to obtain verifiable conditions that guarantee existence, uniqueness, and predictable convergence of fixed points for coupled continuous–discrete dynamics, while embedding the graph geometry directly into the metric via the scaling parameter $\alpha > 0$. Our method studies the coupled operator $T(x, v) = (Bx + \varphi(v), \psi(v))$ and derives explicit sufficient inequalities ensuring that T satisfies a Chatterjea-type contraction on (Z, d_α) , yielding an effective contraction factor $\beta(\alpha)$. In particular, the threshold $\beta(\alpha) < \frac{1}{3}$ implies that T admits a unique fixed point $z^* \in Z$ and that the hybrid Picard iteration converges geometrically in d_α . Numerical experiments support these findings and clarify the integrate mechanism, when ψ maps every vertex to a fixed node, the discrete mode stabilizes after the first iterate, and the successive iterate error $d_\alpha(z_{n+1}, z_n)$ decays exponentially at a rate consistent with $\beta(\alpha)$, with numerical and analytic fixed points agreeing up to floating-point tolerance. Practically, the bound $\beta(\alpha)$ provides an a priori, computable convergence for implementations of matrix graph iterations relevant to graph structured computing and networked models. Future work includes reducing conservatism in the sufficient bounds, exploring richer couplings, and extending the analysis to broader graph classes.

Keywords: Chatterjea contraction; Fixed point; Integrate metric; Matrix induced metric; Weighted graph; Shortest path metric; Picard iteration

INTRODUCTION

Fixed point theory remains a cornerstone in nonlinear analysis for establishing existence, uniqueness, and the convergence of iterative procedures that arise in nonlinear equations, optimization, and discrete dynamical models (Aggarwal et al., 2022; Berinde & Pacurar, 2019). Among the many contractive settings, Chatterjea type conditions are particularly appealing because they replace the classical Lipschitz requirement with a cross-distance inequality of the form $d(Tx, Ty) \leq \alpha(d(x, Ty) + d(y, Tx))$, which supports Picard-type iteration analysis beyond the standard Banach contraction setting. Recent work has continued to refine Chatterjea type inequalities, weaken auxiliary assumptions, and introduce multi-point analogues that retain the characteristic Chatterjea behavior, including possible discontinuity of the underlying map. In particular, generalized Chatterjea type mappings and their fixed point principles have been studied in standard metric spaces with explicit attention to such phenomena (Păcurar & Popescu, 2024). Related modern variants also combine Chatterjea structure with other contraction mechanisms, for instance through large contraction hybrids (Mesmouli et al., 2024). Over the last decade, two complementary research directions have been especially visible. The first develops Chatterjea conditions (e.g., interpolative and Suzuki-flavored forms) designed to enlarge admissible operator classes while preserving strong conclusions. For example, interpolative Chatterjea and cyclic Chatterjea contractions have been investigated in quasi-partial b -metric settings, yielding fixed point results under refined inequalities (Gautam et al., 2021), and further work has explored nonunique fixed point conclusions for interpolative Suzuki Chatterjea type assumptions in the same generalized framework (Gautam et al., 2022). The second direction extends Chatterjea ideas to structured environments, where the geometry

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is not purely metric but constrained by additional relations; one prominent model is a metric space endowed with a directed graph, where contractive requirements are imposed only along admissible edges. Modern developments in this graph aware setting include foundational formulations of graphical metric spaces (Shukla et al., 2017), Chatterjea contractions adapted to graphs (Chaira et al., 2021), (Fallahi & Aghanians, 2016), and broader graphical contraction principles and stability results for metric spaces with graphs (Reich & Zaslavski, 2021). Several refinements also treat graphical contractions in *b*-metric-type environments (Chuensupantharat et al., 2018; De la Sen et al., 2019).

Motivated by the fact that many contemporary applications combine continuous operator dynamics with discrete relational constraints, this paper studies generalized Chatterjea type fixed point theorems on a hybrid matrix graph metric space. On the continuous component, matrix norms are routinely used to control iterates and relate convergence to operator behavior, and recent studies continue to clarify norm-based bounds relevant to iterative matrix analysis and numerical linear algebra (Kittisopaporn et al., 2021; Stoll, 2020). On the discrete component, weighted graphs naturally induce shortest-path distances, and related computable distance structures (Garijo et al., 2023) describe conditions under which edge weights generate consistent (pseudo)metrics and how shortest-path structures behave analytically (Ishiki, 2023).

In this paper, we work on a cartesian product space combining with a matrix norm based metric and a shortest-path metric on a weighted directed graph, coupled through a scaling parameter that balances both contributions. Within this setting, we formulate a Chatterjea type contractive condition adapted to the hybrid metric and prove existence uniqueness of the fixed point together with Picard iteration convergence. Our results connect to generalized Chatterjea theory with graph based fixed point analysis and provide a framework to justify iterative solvers with Picard iteration or related schemes, under simultaneous operator-driven dynamics and graph-structured feasibility constraints, as commonly illustrated in fixed-point-based numerical studies for fractional models and boundary value problems, and fractional type models and in algorithmic treatments on metric spaces with graphs. (Abdou & Alasmari, 2021; Albargi, 2023; Ali et al., 2024) The remainder of the paper is organized as follows. Section 2 collects literature review, including the hybrid metric construction, matrix norm properties, and shortest path distances on weighted graphs. Section 3 presents the main fixed point theorems and illustrative examples. Section 4 concludes with remarks and potential extensions.

LITERATURE REVIEW

Fixed-point theory provides a standard analytic cornerstone for iterative methods in scientific computing. The Banach contraction principle is the canonical tool, it guarantees existence and uniqueness of a fixed point in complete metric spaces and ensures that the Picard iteration converges from any initial condition (Navascués, 2024). Within this tradition, Chatterjea introduced an alternative contractive inequality that is not reducible to the Banach form yet still yields uniqueness and convergence under completeness assumptions (Berinde & Pacurar, 2021).

Recent developments extend Chatterjea type conditions in two complementary directions. First, the inequality has been adapted to broader metric-like settings (e.g., extended *b*-metrics and quasi-partial *b*-metrics, including Suzuki/interpolative variants) to enlarge the admissible class of nonlinear operators while preserving convergence guarantees (Abdou & Alasmari, 2021; Gautam et al., 2021, 2022; Păcurar & Popescu, 2024). Second, the same fixed point ideas have been integrated with iteration schemes motivated by computational problems, including new update rules and convergence analyses arising in nonlinear equations and matrix-based computations (Ali et al., 2024; Kittisopaporn et al., 2021).

A separate line of work enriches metric spaces by incorporating graph structure. Graphical fixed-point frameworks encode admissibility constraints via edges and allow contractive behavior to be required only along graph-related pairs. This approach has been developed as a generalized setting for fixed-point theory and refined for various graphical contractions and *b*-metric-like environments (Chuensupantharat et al., 2018; De la Sen et al., 2019; Reich & Zaslavski, 2021; Shukla et al., 2017). Chatterjea-type contractions have also been studied in metric spaces endowed with a graph, indicating that graph constraints can be compatible with Chatterjea inequalities (Chaira et al., 2021; Fallahi & Aghanians, 2016).

For applications in computer science, hybrid models frequently mix continuous and discrete components. On the continuous side, matrix-induced norms and matrix analysis provide distance measures that reflect scaling and correlation across variables through an invertible linear transformation (Horn & Johnson, 2013). Such matrix-oriented

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viewpoints appear naturally in data-science related matrix methods and in convergence studies of matrix equations (Kittisopaporn et al., 2021; Stoll, 2020). On the discrete side, a weighted graph can be metrized by shortest-path length, a standard construction in graph theory with numerous variants and distance-based descriptors (Diestel, 2006; Garijo et al., 2023). The gap addressed here is an explicit synthesis of these directions: we construct a product metric space that combines a matrix-induced metric with a shortest-path metric on a weighted graph, and we formulate a generalized Chatterjea type contraction compatible with this hybrid structure. The resulting fixed-point theorem and the associated Picard Chatterjea iteration provide a rigorous basis for analyzing computational operators that couple linear transformations with based of graphs.

METHOD

This study is a theoretical and literature based investigation aimed at generalizing the classical Chatterjea contraction to a integrated metric environment. The methodological flow is: (i) construct the integrated metric space, (ii) define the relevant contractive condition, (iii) prove existence and uniqueness of the fixed point under completeness assumptions, and (iv) analyze convergence of the iteration as summarized in Figure 1.

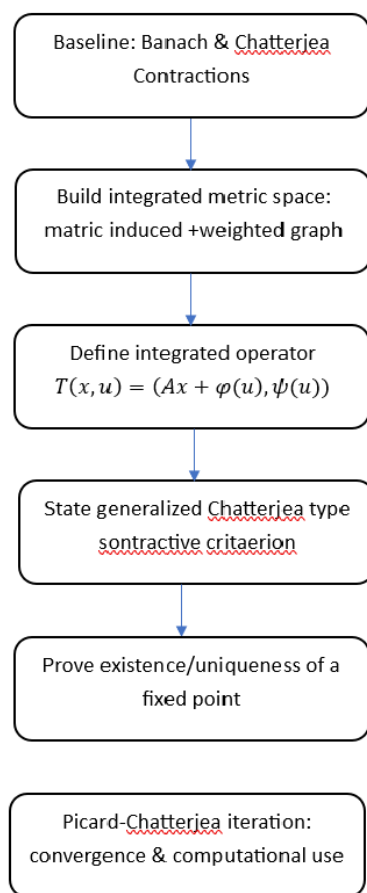


Fig 1. Research workflow.

Throughout this paper, $(Y, \|\cdot\|)$ denotes a real normed linear space, typically $Y = \mathbb{R}^m$. Let $M \in \mathbb{R}^{m \times m}$ be an invertible matrix. For discrete component, let $G = (V, E, w)$ be a connected, undirected weighted graph with a finite vertex set V , an edge set $E \subseteq \{\{u, v\}: u, v \in V, u \neq v\}$, and a positive weight fuction $w: E \rightarrow (0, \infty)$. We consider the hybrid product space $Z = Y \times V$, equipped with an integrated metric d_α , where $\alpha > 0$ is scaling parameter that balances the continuous and discrete contributions.

2.1. Matrix-Induced Metric.

We first recall a standard metric construction on the continuous component.

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Definition 2.1. For $x, y \in Y$, define

$$d_M(x, y) = \|M(x - y)\|, \quad \forall x, y \in Y$$

Lemma 2.1. The function $d_M: Y \times Y \rightarrow [0, \infty)$ is a metric on Y .

Proof. Non-negativity and symmetry follow from properties of the norm. If $d_M(x, y) = 0$, then $\|M(x - y)\| = 0$, hence $M(x - y) = 0$. Since M is invertible, so $x = y$. For the triangle inequality,

$$\begin{aligned} d_M(x, z) &= \|M(x - z)\| = \|M[(x - y) + (y - z)]\| \\ &\leq \|M(x - y)\| + \|M(y - z)\| = d_M(x, y) + d_M(y, z) \end{aligned}$$

Thus M is a metric. ■

Remark 2.1 If $(Y, \|\cdot\|)$ is complete $Y = \mathbb{R}^m$, then (Y, d_M) is complete. Indeed, since M is d_M is equivalent to the norm metric induced by $\|\cdot\|$, and both generate the same notion of Cauchy sequences and convergence.

2.2. Shortest-Path Metric on a Weighted Graph

We next recall the standard metric on a connected weighted graph.

Definition 2.2. Let $G = (V, E, w)$ be a connected undirected weighted graph with $w(e) > 0$ for all $e \in E$. For $u, v \in V$, define

$$d_G(u, v) := \min \left\{ \sum_{i=0}^{k-1} w(\{u_i, u_{i+1}\}) : u = u_0, u_1, \dots, u_k = v \text{ is a path in } G \right\}$$

(The minimum exists since V is finite and $w > 0$.)

Lemma 2.2. The pair (V, d_G) is a metric space.

Proof. Clearly $d_G(u, v) \geq 0$ and $d_G(u, u) = 0$. If $d_G(u, v) = 0$ for $u \neq v$, then there exists a path from u to v with total weight 0, contradicting $t > 0$. Symmetry holds because the graph is undirected, so reversing any path yields the same total weight. For the triangle inequality, concatenate a shortest path from u to v and a shortest path from v to t to obtain a path from u to t ; taking minimum given by: $d_G(u, t) \leq d_G(u, v) + d_G(v, t)$. Thus d_G is a metric. ■

Remark 2.1 . Since V is finite, every Cauchy sequence in (V, d_G) is eventually constant; hence (V, d_G) is complete.

2.3. Integrated Product Space

We now combine the continuous and discrete components into a single analytic framework.

Definition 2.3 Let $\alpha > 0$. For $z_1 = (x, u) \in Z$ and $z_2 = (y, v) \in Z$ define $d_\alpha = (z_1, z_2) = d_M(x, y) + \alpha d_G(u, v)$.

Lemma 2.3. The function d_α is a metric on Z .

Proof. Non-negativity and symmetry are immediate. If $d_\alpha((x, u), (y, v)) = 0$, then $d_M(x, y) = 0$ and $d_G(u, v) = 0$ and $d_M(x, y) = 0$ so $x = y$ and $u = v$. For the triangle inequality, for any $(x, u), (y, v), (z, t) \in Z$,

$$\begin{aligned} d_\alpha((x, u), (z, t)) &= d_M(x, z) + \alpha d_G(u, t) \\ &\leq (d_M(x, y) + d_G(y, z)) + \alpha(d_G(u, v) + d_G(v, t)) \\ &= d_\alpha((x, u), (y, v)) + d_\alpha((y, v), (z, t)) \end{aligned}$$

Hence d_α is a metric. ■

Definition 2.5. A sequence $\{(x_n, u_n)\} \subset Z$ converges to $(x, u) \in Z$ if $d_\alpha((x_n, u_n), (x, u)) \rightarrow 0$. It is Cauchy if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $d_\alpha((x_n, u_n), (x_m, u_m)) < \varepsilon$ for all $m, n \geq N$. The space (Z, d_α) is complete if every Cauchy sequence converges in Z .

Lemma 2.4. If (Y, d_M) is complete and (V, d_G) is complete, then (Z, d_α) is complete.

Proof. Let $\{(x_n, u_n)\}$ be Cauchy in (Z, d_α) . Then $d_M(x_n, x_m) \leq d_\alpha((x_n, u_n), (x_m, u_m))$, so $\{x_n\}$ is Cauchy in (Y, d_M) . Likewise, $d_G(u_n, u_m) \leq \alpha^{-1} d_\alpha((x_n, u_n), (x_m, u_m))$, so $\{u_n\}$ is Cauchy in (V, d_G) . By completeness, $x_n \rightarrow x^*$ in X and $u_n \rightarrow u^*$ in V , hence $(x_n, u_n) \rightarrow (x^*, u^*)$ in Z . ■

2.4. Chatterjea Type Contraction

We recall the classical Chatterjea contraction principle as the conceptual backbone for our integrated extension.

Definition 2.6. Let (M, d) be a metric space. A mapping $T: M \rightarrow M$ is called Chatterjea contraction if there exists $c \in [0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq c(d(x, Ty) + d(y, Tx)), \quad \forall x, y \in M$$

Theorem 2.1 If (M, d) be a complete metric space and T is a Chatterjea contraction then T admits a unique fixed point $x^* \in M$, and the Picard iteration $x_{n+1} = Tx_n$ converges to x^* for initial point $x_0 \in M$.

We also use a practical sufficient condition to verify Chatterjea type contractivity through a Banach estimate.

2.5. From Banach Contraction to Chatterjea Contraction

Lemma 2.5 Let (M, d) be a metric space and suppose $T: M \rightarrow M$ is a Banach contraction:

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$$d(Tx, Ty) \leq k d(x, y), \forall x, y \in M$$

for some $k \in [0, 1)$. If $k < \frac{1}{3}$, then T satisfies the Chatterjea inequality with constant

$$c = \frac{k}{1-2k} \in [0, \frac{1}{2})$$

Proof. By the triangle inequality,

$$d(x, y) \leq d(x, Ty) + d(Ty, Tx) + d(Tx, y) = d(x, Ty) + d(y, Tx) + d(Tx, Ty).$$

Applying the Banach estimate gives $d(Tx, Ty) \leq k d(x, y)$, hence

$$d(Tx, Ty) \leq k(d(x, Ty) + d(y, Tx) + d(Tx, Ty)).$$

Rearranging yields

$$(1 - k)d(Tx, Ty) \leq k(d(x, Ty) + d(y, Tx)), \text{ so we have}$$

$$d(Tx, Ty) \leq \frac{k}{1-k} (d(x, Ty) + d(y, Tx)).$$

If $k < \frac{1}{3}$, then $c = \frac{k}{1-2k} < \frac{1}{2}$ ■

Corollary 2.7. If (M, d) is complete and T is a Banach contraction with $k < \frac{1}{3}$, then T has a unique fixed point and the Picard iteration converges to it by theorem 2.1.

The next section will define the integrated mapping T and establish conditions explicit in α and the operator parameters under which T becomes a Chatterjea type contraction on (Z, d_α) .

RESULT

In This section discusses the combined metric space (Z, d_α) , the fixed point theory for the generalized Chatterjea contraction mapping on (Z, d_α) along with the derivation of sufficient conditions that can be checked from the matrix and graph parameters to guarantee the existence, uniqueness of fixed points, and convergence of iterations.

3.1. Integrated Operator on (Z, d_α)

We study a coupled integrated mapping $T: Z \rightarrow Z$ of the form $T(x, u) = (Bx + \varphi(u), \psi(u))$ $(x, u) \in Z$,

Where $B \in \mathbb{R}^{m \times m}$, $\varphi: V \rightarrow Y$, and $\psi: V \rightarrow V$.

In this paper we twofold approach, (i) Integrated metric coupling. Unlike frameworks where graph is used merely as an auxiliary relation or admissibility structure, we incorporate the graph geometry directly into metric through contractivity and convergence. (ii) explicit verifiable parameter regime for a coupled integrated map. We provide a checkable sufficient condition, expressed in terms of operator bounds for B , Lipschitz type constants for φ and ψ , and the scaling parameter α , ensuring that the integrated operator becomes a Chatterjea contraction and admits a unique fixed point with guaranteed Picard convergence. These two points allow the theory to be used as a stability tool for hybrid models with continuous states and discrete modes.

3.2. Structural Assumptions and Banach-Type Estimate

We impose the following Lipschitz type assumptions.

Assumption 3.1 There exist constants $\mu \geq 0$, $L_\varphi \geq 0$, and $L_\psi \geq 0$ such that for all $x, y \in V$,

$$d_M(Bx, By) \leq \mu d_M(x, y),$$

$$d_M(\varphi(u), \varphi(v)) \leq L_\varphi d_G(u, v),$$

$$d_G(\psi(u), \psi(v)) \leq L_\psi d_G(u, v),$$

Lemma 3.1 Under assumption 3.1, for all $(x, y), (u, v) \in Z$,

$$d_\alpha(T(x, u), T(y, v)) \leq \beta(\alpha) d_\alpha((x, u), (y, v))$$

Where $\beta_\alpha = \max \left\{ \mu, \frac{L_\varphi + \alpha L_\psi}{\alpha} \right\} = \max \left\{ \mu, \frac{L_\varphi}{\alpha} + L_\psi \right\}$

Proof: by definition of d_α ,

$$d_\alpha(T(x, u), T(y, v)) = d_M(Bx + \varphi(u), By + \varphi(v)) + \alpha d_G(\psi(u), \psi(v))$$

Using the triangle inequality for d_M and assumption 3.1

$$d_M(Bx + \varphi(u), By + \varphi(v)) \leq d_M(Bx, By) + d_M(\varphi(u), \varphi(v)) \leq \mu d_M(x, y) + L_\varphi d_G(u, v)$$

Also,

$$\alpha d_G(\psi(u), \psi(v)) \leq \alpha d_G(u, v)$$

Combining

$$d_\alpha(T(x, u), T(y, v)) \leq \mu d_M(x, y) + (L_\varphi + \alpha L_\psi) d_G(u, v) \leq \beta(\alpha) (d_M(x, y) + \alpha d_G(u, v))$$

Which is the desired estimate. ■

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3.3 Chatterjea contractivity and Fixed Point Existence.

We now establish our main fixed point result.

Theorem 3.2. Assume (Y, d_M) is complete and V is finite. Let $T: Z \rightarrow Z$ be defined by $T(x, v) = (Bx + \varphi(v), \psi(v))$

If assumption 3.1 hold and $\beta(\alpha) < \frac{1}{3}$, then T is a Chatterjea contraction on the complete metric space (Z, d_α) .

Consequently, T admits a unique fixed point $z^* = (x^*, v^*) \in Z$, and for every initial point $z_0 \in Z$, the Picard iteration $z_{n+1} = T(z_n)$, $n \geq 0$, converges to z^* in the metric d_α .

Proof. By Lemma 3.1, T is a Banach contraction on (Z, d_α) with constant $\beta(\alpha)$. Since $\beta(\alpha) < 1/3$, lemma 2.5 implies that T is a Chatterjea contraction. Because (Z, d_α) is complete, theorem 2.1 yields existence and uniqueness of the fixed point and convergence of Picard iteration. ■

3.4 Convergence Rate

A direct consequence of the contraction estimate is geometric convergence.

Corollary 3.3. Under the assumption of theorem 3.2, the Picard iterates satisfy

$d_\alpha(z_n, z^*) \leq \beta(\alpha)^n d_\alpha(z_0, z^*)$, $n \geq 0$. In particular, the convergence is at least geometric.

Remark 3.1 The integrated metric provides a quantitative balance between continuous and discrete convergence. For large α , the graph discrete discrepancy is amplified in d_α , which may improve discrete mode stabilization but can tighten that sufficient condition through the term $\frac{L_\varphi}{\alpha} + L_\psi$. This highlights that α is not only a modeling parameter but also a stability tuning parameter.

3.5 Spectral Radius Interpretation

We provide a spectral interpretation of the continuous contraction factor μ . Assume $Y = \mathbb{R}^m$ and $d_M(x, y) = \|M(x - y)\|$ where $\|\cdot\|$ is a vector norm with induced operator norm $\|\cdot\|$. Then

$$d_M(Bx, By) = \|M(Bx - By)\| = \|MB(x - y)\| \leq \|MBM^{-1}\| \cdot \|M(x - y)\| = \|MBM^{-1}\| d_M(x, y).$$

Hence one may take $\mu = \|MBM^{-1}\|$

Moreover, the spectral radius $\rho(B)$ satisfies the classical identity $\rho(B) = \inf_{\|\cdot\|} \|B\|$, where the infimum is taken over

all induced matrix norms. In particular, if $\rho(B) < 1$, then for every $\varepsilon > 0$ there exists an induced norm such that $\|B\| \leq \rho(B) + \varepsilon < 1$. This provide a useful interpretation: the contractivity of the linear component can be viewed as a norm dependent surrogate of spectral stability. However, in practical applications one often work with a fixed norm (e.g., Euclidean norm), and therefore $\mu < 1$ may be stronger than $\rho(B) < 1$ but remains computationally convenient as it avoids explicit eigenvalue computations.

3.6 Picard Iteration Algorithm

For completeness, we state the fixed point iteration procedure.

Algorithm 3.1.

Input : Matrix of B , function φ , also ψ and parameter $\alpha > 0$.

Initial point $z_0 = (x_0, v_0) \in Z$

Tolerance $\varepsilon > 0$, with maximum iteration N_{max}

Initialization : $n = 0$

Iteration : while $n < N_{max}$ do

1. Calculate $p_{n+1} = T(p_n) = (Mx_n + \varphi(u_n), \psi(u_n))$
2. If $d_\alpha(z_{n+1}, z_n) < \varepsilon$, stop and return to (z_{n+1})
3. $n \leftarrow n + 1$

Output : Approximation $(z_{n+1}) \approx z^*$.

To demonstrate that the discrete dynamics can be non trivial while still contractive, we provide a graph example where ψ is not constant but remains Lipschitz with a constant $L_\psi < 1$.

Example 3.1. Let $V = \{1, 2, 3\}$ and consider the weighted graph with edges $E = \{\{1, 2\}, \{2, 3\}, \{1, 3\}\}$, and weights $w(\{1, 2\}) = 1$, $w(\{2, 3\}) = 1$, $w(\{1, 3\}) = 4$,

Then the shortest path metric satisfies: $d_G(1, 2) = 1$, $d_G(2, 3) = 1$, $d_G(1, 3) = 2$

Define a non constant mapping $\psi: V \rightarrow V$ by $\psi(1) = 2$, $\psi(2) = 3$, $\psi(3) = 2$

This map collapses all vertices toward the hub vertex 2, but is not constant on the first step for vertex 1 and 3.

We compute its Lipschitz constant:

- If $u = v$, the inequality holds trivially.
- If $u \neq v$, then $d_G(\psi(u), \psi(v)) = d_G(2, 2) = 0$, hence $d_G(\psi(u), \psi(v)) = 0 \leq L_\psi d_G(u, v)$ for any $L_\psi \in [0, 1)$.

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Thus ψ is Lipschitz with $L_\psi = 0$, and it is contractive in a strong sense while still having non trivial discrete behavior before stabilization.

Now let $Y = \mathbb{R}$, take $M = 1$ so that $d_M(x, y) = |x - y|$, and choose $B = 0.2$. Define $\varphi: V \rightarrow \mathbb{R}$ by

$$\varphi(1) = 1, \quad \varphi(2) = 0 \quad \varphi(3) = -1.$$

Then for any $u, v \in V$, $|\varphi(u) - \varphi(v)| \leq 1 \cdot d_G(u, v)$,

So we may take $L_\varphi = 1$. Also, $d_M(Bx, By) = 0.2|x - y|$, hence $\mu = 0.2$. with scaling $\alpha = 10$, we obtain

$$\beta(\alpha) = \max\left\{0.2, \frac{1}{10} + 0\right\} = 0.2 < \frac{1}{3}$$

Therefore, theorem 3.2 applies: the integrated operator admits a unique fixed point and the Picard iteration converges in the integrated metric.

We now provide an illustrative example combining a matrix-induced metric on $Y \subset \mathbb{R}^2$ with a graph metric on V , highlighting the effectiveness of the integrated distance d_α in guaranteeing fixed-point convergence.

Example 3.2 Let $Y = [0, 1] \times [0, 1] \subset \mathbb{R}^2$ equipped with the matrix induced metric

$$d_M(x, y) = |x_1 - y_1| + 2|x_2 - y_2|, \quad x = (x_1, x_2), y = (y_1, y_2),$$

Which is induced by the invertible matrix

$$M = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \quad \text{via } d_M(x, y) = \|M(x - y)\|$$

Let $G = (V, E, w)$ be the complete graph on three vertices $V = \{v_0, v_1, v_2\}$ endowed with the discrete metric

$$d_G(v_i, v_j) = \begin{cases} 0, & i = j \\ 1, & i \neq j \end{cases}$$

Define $\alpha = 0.1$ and the integrated space $Z = Y \times V$ with the integrated metric

$$d_\alpha((x, u), (y, v)) = d_M(x, y) + \alpha d_G(u, v)$$

Define the integrated mapping $T: Z \rightarrow Z$ by

$$T(x, u) = (Bx + \varphi(u), \psi(u))$$

Where $B = \begin{pmatrix} 0.1 & 0 \\ 0 & 0.1 \end{pmatrix}$, $\varphi(v_i) = (0.1, 0)$ for all i , $\psi(v_i) = v_0$ for all i ,

The following proposition confirms that the mapping T constructed in this Example 3.2 falls within the scope of our main fixed-point theorem, and therefore admits a unique fixed point with guaranteed convergence of the Picard iteration.

Proposition 3.2. The mapping T defined above is a Chatterjea contraction on (Z, d_α) . Consequently, it has a unique fixed point z^* for every initial point $z_0 \in Z$.

Proof. First, for the linear part we estimate the contraction in the d_M - metric space. Since $d_M(Bx, By) = \|MB(x - y)\| = \|MBM^{-1}M(x - y)\| \leq \|MBM^{-1}\|d_M(x, y)$, We can take $\mu = \|MBM^{-1}\|$. Here $MBM^{-1} = 0.1I$, hence $\mu = 0.1$. Next, φ is constant, so $d_M(\varphi(u), \varphi(v)) = 0$, then $L_\varphi = 0$.

Also, ψ is constant, so $d_G(\psi(u), \psi(v)) = 0$, then $L_\psi = 0$.

Therefore the effective Banach contraction factor becomes

$$\beta(\alpha) = \max\left\{\mu, \frac{L_\varphi}{\alpha} + L_\psi\right\} = \max\{0.1, 0\} = 0.1 \leq \frac{1}{3}.$$

By the Banach to Chatterjea in lemma 2.5, T is a Chatterjea contraction on (Z, d_α) . Since (Z, d_α) is complete, T admits a unique fixed point and the Picard converge to it. We now compute the fixed point explicitly and confirm the convergence numerically. Since, $\psi(v) = v_0$ for all $v \in V$, the discrete fixed point must satisfy

$$v^* = \psi(v^*) = v_0$$

For the continuous component, using $\varphi(v_0) = (0.1, 0)$, the fixed point equation $x^* = Bx^* + \varphi(v^*)$ becomes $x^* = 0.1x^* + (0.1, 0)$. Hence $(1 - 0.1)x^* = (0.1, 0)$, and therefore

$$x^* = \left(\frac{0.1}{1 - 0.1}, 0\right) = (0.111111 \dots, 0)$$

Consequently, the unique fixed point of T is $z^* = ((0.111111 \dots, 0), v_0)$.

Starting from $z_0 = (x_0, v_2)$ with $x_0 = (0.374540, 0.950714)$, the Picard iteration $z_{n+1} = T(z_n)$ yields $v_1 = v_0$ and $v_n = v_0$ for all $n \geq 1$, while $x_{n+1} = 0.1x_n + (0.1, 0)$.

In particular, $x_2^{(n)} = 0.1^n x_2^{(0)}$, which explains the very fast decay of the second component observed in table 1.

Table 1

Iteration of fixed point

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Iteration (n)	$x_1^{(n)}$	$x_2^{(n)}$	$v^{(n)}$	Error $d_\alpha(z_{n+1}, z_n)$
0	0.374540	0.950714	v_2	-
1	0.137454	0.095071	v_0	2.05×10^0
5	0.111114	0.000010	v_0	1.95×10^{-4}
10	0.111111	0.000000	v_0	1.95×10^{-9}

DISCUSSIONS

In this case, the fixed-point convergence predicted by theorem 3.2 on the integrated metric space (Z, d_α) , where the integrated state is $z_0 = (x_n, v_n) \in Y \times V$. The discrete component is governed by the graph update $V_{n+1} = \psi(v_n)$. While the continuous component follows the coupled linear rule $x_{n+1} = Bx_n + \varphi(v_n)$. The integrated distance $d_\alpha(z, w) = d_M(x, y) + \alpha d_G(u, v)$ allows both continuous and discrete discrepancies to be tracked simultaneously, which is central to the integrate contraction analysis. As reported in table 1, the discrete state reaches the fixed vertex after the first iteration and remains unchanged thereafter. This behavior is explained by the structure of ψ which maps every vertex to the same target node. Consequently, $d_G(v_{n+1}, v_n) = 0$, for all $n \geq 1$, and the discrete dynamics contributes only at the first step. This feature demonstrates how a graph update can enforce rapid mode stabilization in hybrid models, while the continuous component continues evolving under the influence of the stabilized mode. Related fixed point frameworks incorporating graph structures have been investigated in different settings (Reich & Zaslavski, 2021; Shukla et al., 2017), and Chatterjea-type contractive conditions on graph based spaces have been studied as well (Fallahi & Aghanians, 2016). In contrast, the present example emphasizes a quantitative integration of the graph geometry into the metric itself via d_α , rather than using the graph only as an admissibility structure.

Figure 2 shows the evolution of the continuous components x_1 and x_2 compared with their analytic limits. Both components converge rapidly, with x_2 approaching its limit faster due to stronger damping in the corresponding update component. Figure 3 (phase plot) illustrates the trajectory x_1^n, x_2^n moving toward the analytic fixed point. Such a phase space view complements the error plot by revealing the geometry of the convergence path rather than only its magnitude. The convergence plots confirm the expected geometric behavior of a contractive iteration. Figure 4 shows the successive iterate error $d_\alpha(z_{n+1}, z_n)$ on a logarithmic scale. The nearly linear trend indicates exponential decay, which is consistent with a Banach type estimate $d_\alpha(z_{n+1}, z_n) \leq \beta(\alpha) d_\alpha(z_n, z_{n-1})$, where $\beta(\alpha)$ is the effective contraction factor derived in the main results. In this example, the computed contraction constant is approximately $\beta(\alpha) = 0.1$, and the error decreases from the order of 10^0 to the order of 10^{-9} within ten iterations. This numerical behavior provides concrete evidence that the integrate Picard iteration inherits the classical geometric convergence signature of contraction mappings. Agreement between numerical and analytic fixed points. After convergence, the numerical fixed point $z^* = (x^*, v^*)$ matches the analytic solution obtained from the coupled fixed-point equations $x^* = Bx^* + \varphi(v^*)$ and $v^* = \psi(v^*)$. The discrepancy between the computed and analytic solutions is on the order of 10^{-9} , which is consistent with the chosen stopping tolerance and floating-point arithmetic. This agreement confirms that the observed convergence is aligned with the theoretical fixed-point characterization.

Finally, the use of the matrix-induced metric $d_M(x, y) = \|M(x - y)\|$ provides a flexible way to encode anisotropic weighting of the continuous coordinates, which is natural in matrix and operator analysis (Horn & Johnson, 2013)). This flexibility is relevant in hybrid applications where different continuous variables may have different physical scales or sensitivities. Overall, the main theoretical message of the paper: by combining matrix-weighted continuous distances and graph shortest-path distances into a single metric d_α one obtains a rigorous and computationally verifiable contraction framework with fast and stable convergence.

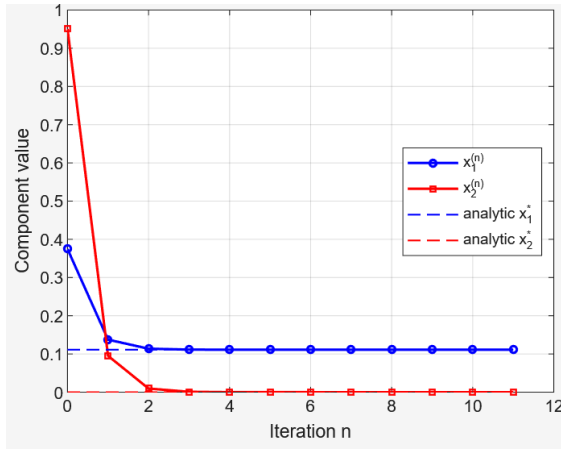


Fig 2. Componentwise convergence

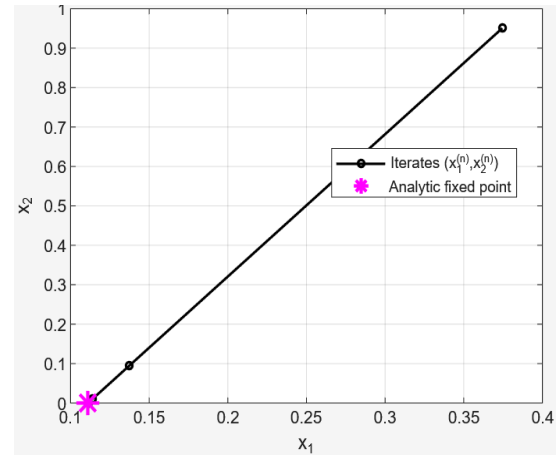


Fig 3. Phase plot of iterates

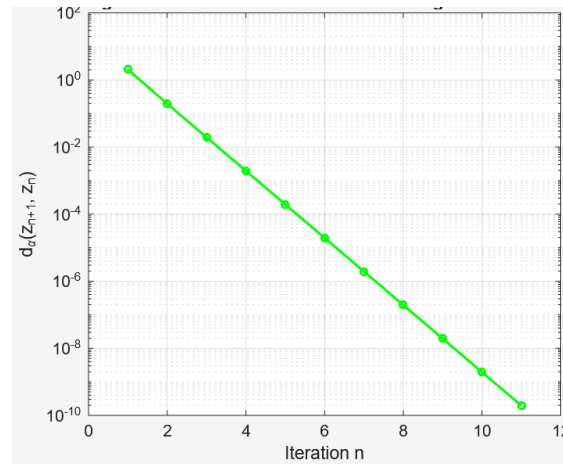


Fig 4. Successive iterate error

CONCLUSION

This paper established an integrate fixed-point framework on the integrated metric space (Z, d_α) , combining a matrix-induced continuous component (Y, d_M) via an invertible matrix M with a graph-based discrete component (Z, d_α) endowed with a shortest-path metric, and investigated the coupled operator $T(x, v) = (Bx + \varphi(v), \psi(v))$ under Chatterjea-type contractive behavior. By deriving an explicit Banach estimate governed by an effective contraction factor $\beta(\alpha)$ and transferring it to a Chatterjea inequality under the verifiable threshold $\beta(\alpha) < \frac{1}{3}$, we proved the existence and uniqueness of a fixed point in Z and obtained geometric convergence of the integrate Picard iteration in the metric d_α , a conclusion supported by numerical experiments showing rapid error decay and stable mode behavior. The main limitations of the present analysis lie in the use of an additive product metric d_α with a single global scaling parameter α , and in global Lipschitz-type assumptions on φ and ψ that can be conservative especially since the sufficient condition $\beta(\alpha) < \frac{1}{3}$ may exclude mappings that satisfy a Chatterjea inequality directly, while the discrete component

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is restricted to finite graphs and the illustrative examples include relatively simple discrete dynamics. Future work will focus on relaxing these sufficient conditions by establishing direct Chatterjea-type bounds (or alternative interpolative variants) on (Z, d_α) , developing systematic sensitivity and tuning strategies for α to optimally balance continuous and discrete contributions, exploring richer couplings such as state-dependent weights $\alpha = \alpha(x, v)$ or non-additive integrate metrics, and extending the theory to broader graph classes (including infinite or random graphs) and operator families motivated by hybrid optimization and control applications.

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